Pearson Edexcel Level 3

GCE Further Mathematics

Advanced Subsidiary

Further Pure Mathematics 1

Specimen paper	**Paper Reference(s)**
Time: 50 minutes	**8FM0/21**

You must have:
Mathematical Formulae and Statistical Tables, calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 5 questions in this section of the paper. The total mark is 40.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Answer ALL questions. Write your answers in the spaces provided.

1. The variable y satisfies the differential equation

$$\frac{d^2y}{dx^2} = 3 + 2y^2\frac{dy}{dx}.$$

Given that $y = 1.5$ and $\frac{dy}{dx} = 2$ at $x = 1$, use the approximations

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2} \text{ and } \left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h},$$

with $h = 0.1$, to find an estimate for y at $x = 0.9$.

(Total for Question 1 is 5 marks)

---

2. Use algebra to find the exact set of values of x for which

$$\frac{3x}{2-x} \leqslant \frac{2}{x^2-4}.$$

(Total for Question 2 is 7 marks)

---

3. The rectangular hyperbola H has Cartesian equation $xy = 15$.

The distinct points $P\left(5p, \frac{3}{p}\right)$ and $Q\left(5q, \frac{3}{q}\right)$ lie on H, where $p \neq 0$ and $q \neq 0$.

(a) Show that an equation of the tangent to H at P is

$$3x + 5p^2y = 30p.$$

(4)

(b) Write down an equation of the tangent to H at Q.

(1)

The tangent to H at the point P and the tangent to H at the point Q intersect at the point R.

(c) Find, as single fractions in their simplest form, the coordinates of R in terms of p and q.

(4)

(Total for Question 3 is 9 marks)

---

4. With respect to the origin O, the points A, B, C and D have coordinates (3, 6, 9), (1 ,5, 7), (2, 3, 8) and $(3 + k, 6, 9 - k)$ respectively, where k is a non-zero constant.

(a) Show that $\overrightarrow{AD}$ is perpendicular to the plane ABC.

(2)

The right triangular prism with face ABC and edge AD has twice the volume of the tetrahedron $OBCD$.

(b) Find the possible values of k.

More than 1 value for k

(7)

(Total for Question 4 is 9 marks)

5.

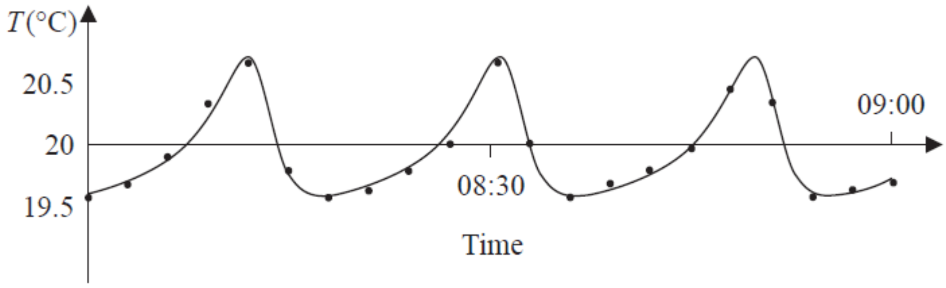


Figure 1

The temperature in a room of a house being regulated by a central heating system was recorded by an engineer every 3 minutes between 08:00 and 09:00 on a particular morning. The temperature outside at 08:00 was recorded as 15 °C.

Using radians, the engineer modelled the temperature, T °C, in the room, x minutes after 08:00 by the equation

$$T = \frac{119 + 38\cos\left(\frac{x}{3}\right) + 79\sin\left(\frac{x}{3}\right)}{6 + 4\sin\left(\frac{x}{3}\right) + 2\cos\left(\frac{x}{3}\right)}.$$

Figure 1 shows the recorded temperatures and the graph resulting from the engineer's model.

Using the t-substitution $t = \tan\dfrac{x}{6}$,

(a) show that equation can be re-written as

$$T = \frac{81t^2 + 158t + 157}{4t^2 + 8t + 8}$$

(3)

The engineer assumes that while the heating system is switched on, the equation will continue to model the temperature beyond 09:00. Given that the heating system remains switched on,

(a) use the answer to part (a) to find the proportion of time that the temperature in the room will be above 20 °C according to the model.

(6)

(b) Give a reason why the equation may not be suitable to model the temperature in the room beyond 09:00.

(1)

(Total for Question 5 is 10 marks)

TOTAL FOR PAPER IS 40 MARKS

1. $x_0 = 1 \quad y_0 = 1.5 \quad \left(\frac{dy}{dx}\right)_0 = 2$

$$\left(\frac{d^2y}{dx^2}\right)_0 = 3 + 2(y_0)^2\left(\frac{dy}{dx}\right)_0$$

$$\left(\frac{d^2y}{dx^2}\right)_0 = 3 + 2(1.5)^2(2) = 12$$

must do 1 approximation, since $h = 0.1$ and $x = 1$.

end goal is to find y_{-1}.

$$y_1 \approx y_{-1} + 2h\left(\frac{dy}{dx}\right)_0$$

$$y_1 = y_{-1} + 2(0.1)(2)$$

$$y_1 = y_{-1} + 0.4$$

$$y_1 = y_{-1} + 0.4 \quad ①$$

$$y_1 \approx 2y_0 - y_{-1} + h^2\left(\frac{d^2y}{dx^2}\right)_0$$

$$y_1 \approx 2(1.5) - y_{-1} + (0.1)^2(12)$$

$$y_1 \approx 3.12 - y_{-1} \quad ②$$

solve ① and ② simultaneously for y_{-1}.

$$y_1 = y_{-1} + 0.4$$

$$y_1 = 3.12 - y_{-1}$$

$$y_{-1} + 0.4 = 3.12 - y_{-1}$$

$$2y_{-1} = 2.72$$

$$y_{-1} = 1.36$$

$y = 1.36$ //

2. $\frac{3x}{2-x} \leq \frac{2}{x^2-4}$

$\frac{-3x}{x-2} \leq \frac{2}{x^2-4}$

$0 \leq \frac{2}{(x+2)(x-2)} + \frac{3x}{(x-2)}$

$0 \leq \frac{2+3x(x+2)}{(x+2)(x-2)}$

$0 \leq \frac{3x^2+6x+2}{(x+2)(x-2)}$

we cannot do

$0 \times (x+2)(x-2) \leq \frac{3x^2+6x+2}{(x+2)(x-2)} \times (x+2)(x-2)$

This is because we are unsure if x is -ve. or +ve.
If x is -ve. , when multiplying both sides by $(x+2)(x-2)$, the $\leq$ would have to flip to $\geq$.
However if we multiply both sides by $(x+2)^2(x-2)^2$ this is a guaranteed +ve. number. ∴ no need for inequality to flip.

$0 \times (x+2)^2(x-2)^2 \leq \frac{3x^2+6x+2}{\cancel{(x+2)}\cancel{(x-2)}} \times (x+2)^{\cancel{2}}(x-2)^{\cancel{2}}$

$0 \leq (3x^2+6x+2)(x+2)(x-2)$

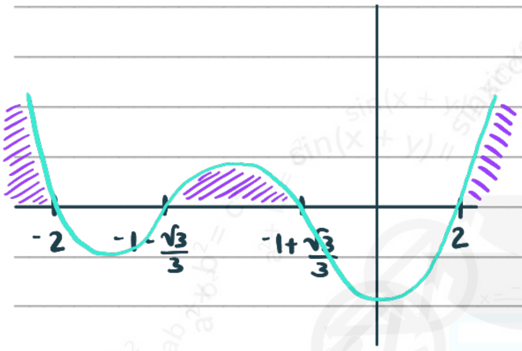
↓ find x-intersects

cannot factorise so solve for + via quadratic formulae

$3x^2+6x+2=0$ solve via quadratic formulae

$$x=\frac{-(6)\pm\sqrt{(6)^2-(4)(3)(2)}}{2(3)}=\frac{-6\pm\sqrt{12}}{6}=\frac{-6\pm2\sqrt{3}}{6}=-1\pm\frac{\sqrt{3}}{3}$$

$$x=-1+\frac{\sqrt{3}}{3},\ -1-\frac{\sqrt{3}}{3},\ -2,\ 2$$



since inequality faces ≥ 0 look above x-axis

$$x<-2,\quad -1-\frac{\sqrt{3}}{3}\leq x\leq -1+\frac{\sqrt{3}}{3},\quad x\geq 2$$

This is not correct - general rule is to sub in critical values and check if they satisfy inequality for ($\geq$ or $\leq$)
don't need to check for ($>$ or $<$)

(1) $x=-2$

$$\frac{3(-2)}{2-(-2)}\not\leq\frac{2}{(-2)^2-4}$$

RHS is undefined so inequality not true for $x=-2$

(2) $x=-1-\frac{\sqrt{3}}{3}$

$$\frac{3\left(-1-\frac{\sqrt{3}}{3}\right)}{2-\left(-1-\frac{\sqrt{3}}{3}\right)}\leq\frac{2}{\left(-1-\frac{\sqrt{3}}{3}\right)^2-4}$$

inequality satisfied

$$\frac{-12+3\sqrt{3}}{13}\leq\frac{-12+3\sqrt{3}}{13}\ \checkmark$$

(3) $x = -1 + \frac{\sqrt{3}}{3}$

$$\frac{3\left(-1+\frac{\sqrt{3}}{3}\right)}{2-\left(-1+\frac{\sqrt{3}}{3}\right)} \leq \frac{2}{\left(-1+\frac{\sqrt{3}}{3}\right)^2-4}$$

inequality satisfied

$$\frac{-12+3\sqrt{3}}{13} \leq \frac{-12+3\sqrt{3}}{13} \checkmark$$

(4) $x = 2$

$$\frac{3(2)}{2-(2)} \nleq \frac{2}{(2)^2-4}$$

LHS and RHS undefined so inequality not true for $x = 2$

$\therefore \{x \in \mathbb{R} : x < -2\} \cup \{x \in \mathbb{R} : -1-\frac{\sqrt{3}}{3} < x < -1+\frac{\sqrt{3}}{3}\} \cup \{x \in \mathbb{R} : x > 2\}$ //

3a. $xy = 15$

$y = \frac{15}{x} = 15x^{-1}$

$\frac{dy}{dx} = -15x^{-2} = \frac{-15}{x^2}$

differentiate w.r.t. x

$\left.\frac{dy}{dx}\right|_{x=5p} \quad \frac{-15}{(5p)^2} = \frac{-15}{25p^2} = -\frac{3}{5p^2}$

sub in x-coord to find gradient @ P.

$M_{tangent} = \frac{-3}{5p^2}$

$y - \left(\frac{3}{p}\right) = \frac{-3}{5p^2}(x - 5p)$

Line eq^n formulae

$y - y_1 = m(x - x_1)$

where (x_1, y_1) known coord. on line

m is gradient

$5p^2y - 15p = -3(x - 5p)$

$5p^2y - 15p = -3x + 15p$

$3x + 5p^2y = 30p$ // (shown)

b. replace p with q from eq^n above.

$3x + 5q^2y = 30q$ //

c. equate both eq^{n}s to find R.

P: $3x = 30p - 5p^2y$

Q: $3x = 30q - 5q^2y$

$30p - 5p^2y = 30q - 5q^2y$

$5q^2y - 5p^2y = 30q - 30p$ $\div 5$

$q^2y - p^2y = 6q - 6p$

$y(q^2 - p^2) = 6(q - p)$

difference of 2 squares

$y\cancel{(q-p)}(q+p) = 6\cancel{(q-p)}$

$y(q+p) = 6$

$y = \frac{6}{q+p}$

(sub back into either tangent eq^n to solve for x

$3x + 5q^2y = 30q$

$3x + 5q^2\left(\frac{6}{q+p}\right) = 30q$

$3x = 30q - \frac{30q^2}{q+p}$

$\div 3$

$x = 10q - \frac{10q^2}{q+p}$

$x = \frac{10q(q+p) - 10q^2}{q+p} = \frac{\cancel{10q^2} + 10qp - \cancel{10q^2}}{q+p} = \frac{10qp}{q+p}$

$x = \frac{10pq}{p+q}$

$R\left(\frac{10pq}{p+q}, \frac{6}{p+q}\right)$ //

4. Firstly find Π eqⁿ ABC.

$r = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$\overrightarrow{AB} = \begin{pmatrix}-3\\-6\\-9\end{pmatrix} + \begin{pmatrix}1\\5\\7\end{pmatrix} = \begin{pmatrix}-2\\-1\\-2\end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$\overrightarrow{AC} = \begin{pmatrix}-3\\-6\\-9\end{pmatrix} + \begin{pmatrix}2\\3\\8\end{pmatrix} = \begin{pmatrix}-1\\-3\\-1\end{pmatrix}$

Π eqⁿ : $r = \begin{pmatrix}3\\6\\9\end{pmatrix} + \lambda\begin{pmatrix}-2\\-1\\-2\end{pmatrix} + \mu\begin{pmatrix}-1\\-3\\-1\end{pmatrix}$

to prove $\overrightarrow{AD}$ is perpendicular, $\overrightarrow{AD}\cdot\overrightarrow{AB} = 0$ and $\overrightarrow{AD}\cdot\overrightarrow{AC} = 0$

$\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD}$

$\overrightarrow{AD} = \begin{pmatrix}-3\\-6\\-9\end{pmatrix} + \begin{pmatrix}3+k\\6\\9-k\end{pmatrix} = \begin{pmatrix}k\\0\\-k\end{pmatrix}$

dot product : $\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}d\\e\\f\end{pmatrix} = (a)(d) + (b)(e) + (c)(f)$

$\overrightarrow{AD}\cdot\overrightarrow{AB} = \begin{pmatrix}-2\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}k\\0\\-k\end{pmatrix} = (-2)(k) + (-1)(0) + (-2)(-k)$

$= -2k + 2k$

$= 0$ //

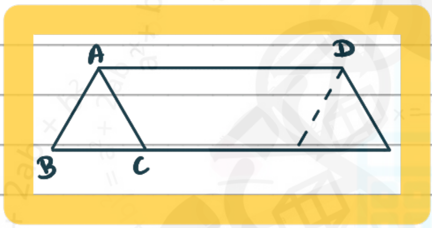
$$\overrightarrow{AD} \cdot \overrightarrow{AC} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix} = (-1)(k) + (-3)(0) + (-1)(-k)$$

$$= -k + k$$

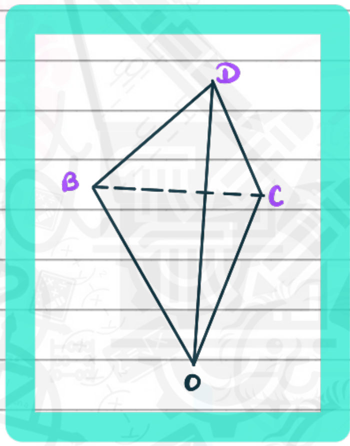
$$= 0 \,//$$

$\therefore \overrightarrow{AD}$ is perpendicular to $\Pi\, ABC \,//$

b.



$= 2 \times$



vol. prism $ABCD$ = $\Delta ABC \times |AD|$

$\Delta ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$

use part (a) and perform cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$$

$$\begin{array}{ccc} i & j & k \\ \hline -2 & -1 & -2 \\ -1 & -3 & -1 \end{array} = i \begin{vmatrix} -1 & -2 \\ -3 & -1 \end{vmatrix} - j \begin{vmatrix} -2 & -2 \\ -1 & -1 \end{vmatrix} + k \begin{vmatrix} -2 & -1 \\ -1 & -3 \end{vmatrix}$$

cross product given in formulae booklet

$= i\left[(-1)(-1)-(-2)(-3)\right] - j\left[(-2)(-1)-(-2)(-1)\right] + k\left[(-2)(-3)-(-1)(-1)\right]$

$\vec{AB} \times \vec{AC} = -5i + 0j + 5k$

$\Delta ABC = \frac{1}{2}\left|\begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}\right|$

$= \frac{1}{2}\sqrt{(5)^2+(0)^2+(5)^2}$

$\Delta ABC = \frac{\sqrt{50}}{2}$

$\Delta ABC \times |\vec{AD}|$

$= \frac{\sqrt{50}}{2} \times \sqrt{(k)^2+(0)^2+(k)^2}$

$= \frac{\sqrt{50}}{2} \times \sqrt{2k^2}$

$= \left|\frac{\sqrt{50}}{2} \times \sqrt{2}\, k\right|$

$= \left|\frac{\sqrt{100}}{2}\, k\right|$

$= \left|\frac{10}{2} k\right|$

$= |5k|$ //

put a modulus sign since k could be -v.e.

Vol. OBCD = $\frac{1}{6}\left|\vec{OD}\cdot(\vec{OB}\times\vec{OC})\right|$

$\vec{OB}\times\vec{OC} = \begin{pmatrix}1\\5\\7\end{pmatrix}\times\begin{pmatrix}2\\3\\8\end{pmatrix}$

$$\begin{array}{ccc} i & j & k \\ \hline 1 & 5 & 7 \\ 2 & 3 & 8 \end{array} = i\begin{vmatrix}5 & 7\\3 & 8\end{vmatrix} - j\begin{vmatrix}1 & 7\\2 & 8\end{vmatrix} + k\begin{vmatrix}1 & 5\\2 & 3\end{vmatrix}$$

Cross product given in F.B.

$= i\left[(5)(8)-(3)(7)\right] - j\left[(1)(8)-(2)(7)\right] + k\left[(1)(3)-(2)(5)\right]$

$= 19i + 6j - 7k$

Vol. OBCD = $\frac{1}{6}\left|\begin{pmatrix}3+k\\6\\9-k\end{pmatrix}\cdot\begin{pmatrix}19\\6\\-7\end{pmatrix}\right|$

dot product: $\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}d\\e\\f\end{pmatrix} = (a)(d)+(b)(e)+(c)(f)$

$= \frac{1}{6}\left|(19)(3+k)+(6)(6)+(9-k)(-7)\right|$

$= \frac{1}{6}\left|57+19k+36-63+7k\right|$

Vol. OBCD = $\frac{1}{6}\left|26k+30\right|$

$\boxed{\text{Vol. } OABC = 2\times \text{Vol. } OBCD}$

$|5k| = 2\times\frac{1}{6}|26k+30|$

$|5k| = \frac{1}{3}|26k + 30|$

$|5k| = \left|\frac{26k}{3} + 10\right|$

To get rid of modulus on both sides, square both sides

$(5k)^2 = \left(\frac{26}{3}k + 10\right)^2$

$25k^2 = \frac{676}{9}k^2 + \frac{520}{3}k + 100$

$\frac{451}{9}k^2 + \frac{520}{3}k + 100 = 0$

$\times 9$

$451k^2 + 1560k + 900 = 0$

solve for k via quadratic formulae

$$k = \frac{-(1560) \pm \sqrt{(1560)^2 - (4)(451)(900)}}{2(451)} = -\frac{30}{41} \text{ or } -\frac{30}{11}$$

$k = -30/11$ or $k = -30/41$ //

5a. given $t = \tan\left(\frac{x}{6}\right)$

use $t = \tan\left(\frac{\theta}{2}\right)$ where $\theta = \frac{x}{3}$ [$t = \tan\left(\frac{\theta}{2}\right)$ is more familiar]

$t = \tan\left(\frac{\theta}{2}\right)$ $\sin(\theta) = \dfrac{2t}{1+t^2}$

$\cos(\theta) = \dfrac{1-t^2}{1+t^2}$

proof @ end of Q

$\sin\left(\frac{x}{3}\right) = \dfrac{2t}{1+t^2}$

$\cos\left(\frac{x}{3}\right) = \dfrac{1-t^2}{1+t^2}$

$$T = \frac{119 + 38\left(\frac{1-t^2}{1+t^2}\right) + 79\left(\frac{2t}{1+t^2}\right)}{6 + 4\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right)}$$

$$T = \frac{\dfrac{119(1+t^2) + 38(1-t^2) + 79(2t)}{\cancel{1+t^2}}}{\dfrac{6(1+t^2) + 4(2t) + 2(1-t^2)}{\cancel{1+t^2}}}$$

$$T = \frac{119(1+t^2) + 38(1-t^2) + 79(2t)}{6(1+t^2) + 4(2t) + 2(1-t^2)} = \frac{119 + 119t^2 + 38 - 38t^2 + 158t}{6 + 6t^2 + 8t + 2 - 2t^2} = \frac{81t^2 + 158t + 157}{4t^2 + 8t + 8}$$

$$\therefore T = \frac{81t^2 + 158t + 157}{4t^2 + 8t + 8} \quad //$$

(Shown)

b. firstly find when t when $T = 20$

$$20 = \frac{81t^2 + 158t + 157}{4t^2 + 8t + 8}$$

$20(4t^2 + 8t + 8) = 81t^2 + 158t + 157$

$80t^2 + 160t + 160 = 81t^2 + 158t + 157$

$0 = t^2 - 2t - 3$

$0 = (t-3)(t+1)$

$t = 3$ or $t = -1$

use substitution $t = \tan\left(\frac{x}{6}\right)$

$\frac{x}{6} = \arctan(t)$

$\frac{x}{6} = \arctan(3)^c$

$\frac{x}{6} = 1.249045772^c, 4.390638426^c, 7.532223108^c, \ldots$

$x = 7.494274634^c, 26.34383056^c, 45.19338648^c, \ldots$

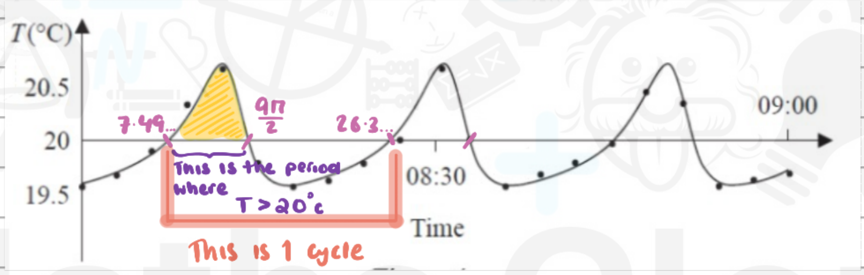
$\frac{x}{6} = \arctan(-1)^c$

$\frac{x}{6} = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

$x = \cancel{-\frac{3\pi}{2}}, \frac{9\pi}{2}, \frac{21\pi}{2}$

$14.137\ldots$

-ve. sol^n so disregard



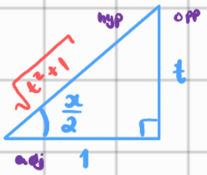
$$\frac{\frac{9\pi}{2} - 7.494274634}{26.34383056 - 7.494274634} \times 100 = 35.241638232$$

(finding proportion in 1 cycle as all cycles are identical)

35.2% (3s.f.)

c. cannot extrapolate using 1hr worth of data

Deriving the t-formulae :



1) draw a right-angled triangle and label angle and sides which you know.

✱ you are allowed to memorise the t-formulae for sin(x), cos(x), tan(x) and do not have to derive it in the exam unless specifically asked.

2) work out hypotenuse in terms of t, (pythagoras)

$$\sqrt{(t)^2+(1)^2}$$
$$= \sqrt{t^2+1}$$

3) label each side of triangle opposite, adjacent, hypotenuse

4) write out $\sin\left(\frac{x}{2}\right)$, $\cos\left(\frac{x}{2}\right)$, $\tan\left(\frac{x}{2}\right)$ in terms of t.

$$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{t^2+1}} \qquad \sin = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{t^2+1}} \qquad \cos = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{t}{1} = t \qquad \tan = \frac{\text{opposite}}{\text{adjacent}} \Big/ \frac{\sin}{\cos}$$

5) Now use double-angle formulae and write in terms of t :

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$
$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\sin(x) = 2\left(\frac{t}{\sqrt{t^2+1}}\right)\left(\frac{1}{\sqrt{t^2+1}}\right) \qquad \cos(x) = \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2$$

$$\sin(x) = \frac{2t}{t^2+1} \qquad \cos(x) = \frac{1-t^2}{1+t^2}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{2t}{\cancel{1+t^2}}}{\frac{1-t^2}{\cancel{1+t^2}}} = \frac{2t}{1-t^2}$$

$$\tan(x) = \frac{2t}{1-t^2}$$